3.1 Means and variances

The Poisson distribution has

\[ p(n) = e^{-\mu} \frac{\mu^n}{n!}. \]

The mean is therefore

\[ m = \sum_{n=0}^{\infty} n \, p(n) \]
\[ = \sum e^{-\mu} \frac{\mu^n}{(n-1)!} \]
\[ = \mu e^{-\mu} \sum \frac{\mu^n}{n!} \]

and the last term sums to \( e^\mu \).

A similar argument works for the variance.

For a power law, the upper and lower limits matter; suppose they are \( b \) and \( a \). Within this range, the power law distribution is

\[ p(x) = \frac{\gamma - 1}{a^{1-\gamma} - b^{1-\gamma}} x^{-\gamma} \]

and the mean is

\[ m = \frac{(-a^\gamma b^2 + a^2 b^\gamma)(-1 + \gamma)}{(-a^\gamma b + a b^\gamma)(-2 + \gamma)} \]

which is quite a complicated expression. It becomes clearer in a few limits. If \( \gamma = 3 \), for instance, we have

\[ m = 2a \]

showing that the bottom end determines the mean. If we have \( \gamma = 2 \), we can take limits to get

\[ m = \frac{ab(\log[a] - \log[b])}{a - b} \]

showing that both \( a \) and \( b \) matter here. If \( \gamma \to 1 \) we get

\[ m = \frac{a - b}{\log[a] - \log[b]} \]

and \( b \) dominates the mean.

The variance is intimidating:
\[
-((1 + \gamma)(-a^{2\gamma}b^4 - a^{4\gamma}b^2 + a^{3+\gamma}b^{1+\gamma}(-2 + \gamma)^2 + a^{1+\gamma}b^{3+\gamma}(-2 + \gamma)^2 - 2a^{2+\gamma}b^{2+\gamma}(3 - 4\gamma + \gamma^2)))/(a^n b - ab^\gamma)^2(-3 + \gamma)(-2 + \gamma)^2)
\]

Taking some limits, we find (as expected) that we must have \(\gamma < -4\) for the upper limit not to dominate the variance; also, we must have \(\gamma > -1/2\) for the lower limit not to matter.

The variance of a Cauchy distribution is simple by comparison.

\[
p(x) = \frac{1}{\pi(1 + (x - \mu)^2)}
\]

and clearly

\[
\int (x - \mu)^2 p(x)
\]

diverges linearly.