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A METHOD FOR DETERMINING LUMINOSITY FUNCTIONS INCORPORATING BOTH FLUX MEASUREMENTS AND FLUX UPPER LIMITS, WITH APPLICATIONS TO THE AVERAGE X-RAY TO OPTICAL LUMINOSITY RATIO FOR QUASARS[1]

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ABSTRACT

A novel method for determining luminosity functions from preselected random samples is derived. The method incorporates positive flux measurements as well as flux upper limits. It is nonparametric, and enables one to determine the best estimate and ranges of uncertainty for the shape of the luminosity function, and for any physical quantity that depends on the shape of the luminosity function. By applying this method to preliminary results from the quasar-survey of the *Einstein* observatory, we determine the distribution of the values of $L_x/L_{\rm opt}$ for quasars, and the average of this ratio. The method is very general and can be used to determine any type of distribution function (not only luminosity functions). It can also be formulated as a parametric method, and be used to determine the parameters of distribution functions with presumed functional forms.

Subject headings: luminosity function — quasars — X-rays: background — X-rays: sources

I. INTRODUCTION

Standard procedures for determining luminosity functions involve observations of complete samples. These samples include all objects of a given population within a specified region of sky, subject to a well-defined set of observational selection criteria. Most commonly used are flux-limited samples, in which one utilizes only those objects whose fluxes are higher than the sample thresholds.

The main purpose of this paper is to derive a method for determining luminosity functions, a method based on an entirely different observational program. A preselected random sample of objects is drawn. Each of the objects in the sample is then observed down to a certain limiting threshold (which can vary from one object to another), thus yielding a measurement of the flux from that object, or an upper limit for the flux. The set of observed fluxes and upper limits is analyzed using a maximum likelihood approach to determine the shape of the luminosity function.

Our method makes it possible to combine large bodies of data acquired with very different sensitivities and, in particular, observations of single objects. The method utilizes both measured fluxes and flux upper limits, and thus makes use also of objects whose fluxes fall below the observational thresholds. For a wide range of circumstances the best estimate for the shape of the luminosity function can be calculated from a simple analytic expression. In other cases, very simple numerical computations are required. Our technique can be formulated either as a nonparametric method, in which case the shape of the luminosity function is determined without any prior assumptions, or as a parametric method, when the parameters of a presumed functional form are determined. In either case the method makes it possible to calculate not only the best estimate, but also ranges of uncertainty for the shape of the luminosity function and for any physical quantity that depends on this shape. The normalization of the luminosity function is not determinable by our method.

We apply our method to preliminary results from the quasar survey undertaken with the *Einstein* observatory (Giacconi *et al.* 1979*a*; Tananbaum *et al.* 1979). We find the shape of the distribution of the values of $L_x/L_{\rm opt}$ nonparametrically, i.e., without any prior assumption on this shape. We also find the best estimate and the range of uncertainty of the average $L_x/L_{\rm opt}$ ratio for quasars. This is the fundamental quantity that enters the calculation of the contribution of quasars to the extragalactic X-ray background, when one uses optical number counts of quasars or assumes the optical luminosity-function of quasars and its cosmological evolution. Our results were used by Tananbaum *et al.* (1979) to discuss the relation of quasars to the diffuse X-ray background.

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The plan of the paper is as follows. In § II we summarize the basic data, obtained with the *Einstein* observatory, that serve as the input for calculating the average $L_x/L_{\rm opt}$ ratio for quasars. This serves two purposes. (*a*) It provides the reader with an explicit example, in order to make it easier to follow the derivation of the general method. (*b*) It states in more detail the assumptions entering the calculation of $\langle L_x/L_{\rm opt}\rangle$, since the results may still be subject to systematic biases. In § III we derive our method and describe in detail its simplest version. This version concerns a luminosity function of one independent variable, the range of values of this variable is binned, and the method is formulated nonparametrically. This is the main part of the paper. In § IV we present the determination of the preliminary estimate of $\langle L_x/L_{\rm opt}\rangle$ for quasars. In § V we describe several generalizations of the method. This section is formulated a bit more formally than the rest of the paper. It is intended for readers who are actually interested in applying the method to more complicated situations, where the simplest version is not adequate. We summarize our results briefly in § VI.

II. BASIC QUASAR DATA

Thirty-five known quasars, previously undetected as X-ray sources (except for 3C 273), were observed with the *Einstein* observatory (see Tananbaum *et al.* 1979 for details). The length of observation for each quasar was determined *a priori*, defining a limiting detectable X-ray flux in each case. Twenty-seven quasars were detected at flux levels above their respective thresholds, whereas eight flux upper limits were obtained for the remaining quasars. The measured fluxes yielded monochromatic X-ray luminosities at 2 keV at the source, $L_{2\,\rm keV}$. Flux upper limits yielded similarly upper limits for $L_{2\,\rm keV}$. Optical monochromatic luminosities at 2500 Å at the source, $L_{2500\,\text{Å}}$, were calculated from optical data. The ratio of X-ray to optical monochromatic luminosities is parametrized by an energy power-law index $\alpha_{0,x}$ so that by using the appropriate numerical values one has

$$L_{2\,\rm keV}/L_{2500\,\text{Å}} = \text{dex}\,(-2.605\alpha_{0,x})\,. \qquad (1)$$

The *Einstein* X-ray observations provide 27 measured values of $\alpha_{0,x}$, and eight lower limits for $\alpha_{0,x}$. A histogram showing the distribution of observed values, binned into intervals of $\Delta\alpha_{0,x} = 0.1$, and the positions of the lower limits are given in Figure 1.

Within the limitations of this preliminary, rather small sample, the observed values of $\alpha_{0,x}$ did not exhibit any obvious strong correlations with $L_{2500\,\text{Å}}$ or z. We therefore make the preliminary assumption that the distribution of the values of $\alpha_{0,x}$ is independent of z, $L_{2500\,\text{Å}}$, or any other intrinsic property of quasars not related to their X-ray luminosity. This is formally equivalent to assuming that the quasar luminosity function $\psi(z, L_{2500\,\text{Å}}, \ldots, L_{2\,\rm keV})$ can be factorized in the form $\phi(z, L_{2500\,\text{Å}}, \ldots)f(\alpha_{0,x})$, where the dots symbolize all additional intrinsic properties not related to $L_{2\,\rm keV}$, and f can be taken to be normalized, $\int d\alpha_{0,x}f(\alpha_{0,x}) = 1$.

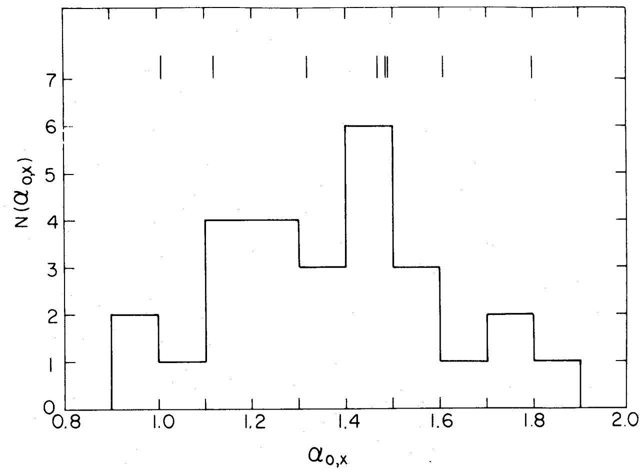

FIG. 1.—A histogram of the values of $\alpha_{0,x}$ for the detected quasars, and the values of the lower limits of $\alpha_{0,x}$ for the undetected quasars (marked by horizontal bars), from the preliminary quasar sample.

Since the 35 quasars were preselected for the X-ray observations without any regard to their X-ray luminosities, the sample can be considered as a preselected random sample concerning the distribution of $\alpha_{0,x}$. Each quasar is an independent probe of the luminosity function $f(\alpha_{0,x})$. A given quasar can be detected if its value of $\alpha_{0,x}$ is smaller than a certain predetermined limiting value $\alpha_{0,x}{}^m$ (which can vary from quasar to quasar). If it is detected, one records the value of $\alpha_{0,x}$. If it is not detected, one records $\alpha_{0,x}{}^m$, which is a lower limit to the actual value of its $\alpha_{0,x}$. Given the set of values of $\alpha_{0,x}$ for the detected quasars and of $\alpha_{0,x}{}^m$ for the undetected quasars, we wish to estimate the function $f(\alpha_{0,x})$, and the corresponding average of $L_{2\,\rm keV}/L_{2500\,\text{Å}}$, given (see eq. [1]) by

$$\langle L_{2\,\rm keV}/L_{2500\,\text{Å}}\rangle = \int d\alpha_{0,x}f(\alpha_{0,x}) \times \text{dex}\,(-2.605\alpha_{0,x})\,. \qquad (2)$$

This can be achieved by our method, described in the next section.

Our preliminary basic assumption is a rather strong one. Consequently, our results and the subsequent conclusions of Tananbaum *et al.* (1979) may be subject to serious systematic biases. The sample is very heavily weighted by radio-observed quasars. If a correlation exists between the quasar radio-luminosity and $L_{2\,\rm keV}$, the sample is biased toward smaller values of $\alpha_{0,x}$. This possibility will be studied by further observations currently in progress. We also note that the X-ray observations yielding $L_{2\,\rm keV}$ were not performed simultaneously with the optical observations that yielded $L_{2500\,\text{Å}}$. Thus the distribution of $\alpha_{0,x}$ may be affected by time variability.

Our results are *not* subject to any statistical bias, which might *a priori* result from the fact that it is easier to observe in X-rays quasars with smaller values of $\alpha_{0,x}$ (cf. Giacconi *et al.* 1979*b*). This is achieved by incorporating into the analysis both the observed values of $\alpha_{0,x}$ and the lower limits to $\alpha_{0,x}$, and by excluding from the analysis quasars selected (discovered) by their X-ray emission.

We note that if the preliminary quasar sample had indicated no correlation at all between $L_{2\,\mathrm{keV}}$ and $L_{2500\,\mathrm{Å}}$, we could have used our method to estimate an X-ray luminosity function of quasars of the form $f(L_{2\,\mathrm{keV}})$ rather than of the form $f(L_{2\,\mathrm{keV}}/L_{2500\,\mathrm{Å}})$.

III. THE METHOD

In this section we describe the basis elements of our method. We present its simplest version, where a distribution function f of one variable α is estimated, the range of values of α is binned, and the estimation is done without presuming any functional form for f (i.e., nonparametric approach).

a) *Notation*

A random variable α is described by a normalized probability distribution function $f(\alpha)$. A sample of J independent values of α is drawn. Let us denote the different drawings by the index j, and the corresponding values of α by α_j. A set of J "thresholds" $\alpha_j^{(m)}$ is specified in advance. If, for some j, $\alpha_j \leq \alpha_j^{(m)}$, then the actual value of α_j is known. If, however, $\alpha_j > \alpha_j^{(m)}$, then the actual value of α_j is not known, but the fact that $\alpha_j > \alpha_j^{(m)}$ is known.

Let us denote by $\tilde{J}$ the number of drawings in which α_j turned out to be $\leq \alpha_j^{(m)}$. For notational simplicity assume that these drawings are labeled by values of j from 1 to $\tilde{J}$. In each of the remaining $J - \tilde{J}$ drawings, labeled by values of j from $\tilde{J} + 1$ to J, α_j turned out to be $> \alpha_j^{(m)}$ and is not known.

We now bin the range of values of the variable α. Suppose there are M bins, labeled by an index k that runs from 1 to M.[5] We let the central value of α in each bin be denoted by $\bar{\alpha}_k$, and we assume that the label k is ordered so that $\bar{\alpha}_1 < \bar{\alpha}_2 < \cdots < \bar{\alpha}_M$.

Finally, let us denote by $k(j)$ the index of the bin inside which the value of α_j falls. Let $k^{(m)}(j)$ denote the index of the bin whose upper boundary is closest to the value of $\alpha_j^{(m)}$. Let us denote the *a priori* probability that α is inside bin k by f_k; this probability is equal to the integral of $f(\alpha)$ inside bin k.

Using the above definitions the sample yields the following data. In $\tilde{J}$ drawings labeled by $1 \leq j \leq \tilde{J}$ the values of α_j are known and fall into the bins whose indices are $k(j)$. In $J - \tilde{J}$ drawings labeled by $\tilde{J} + 1 \leq j \leq J$ the values of α_j are not known, but it is known that each α_j must fall into one of the bins labeled by $k \geq k^{(m)}(j) + 1$ (rounded off to the nearest half bin).

b) *The Likelihood Function*

The *a priori* probability that in drawing j the value of α_j is smaller than its threshold $\alpha_j^{(m)}$ and that α_j falls into bin $k(j)$ is $f_{k(j)}$. The *a priori* probability that in drawing j the value of α_j is larger than $\alpha_j^{(m)}$ and that consequently α_j falls into one of the bins with $k \geq k^{(m)}(j) + 1$ is

$$\sum_{k=k^{(m)}(j)+1}^{M} f_k \,.$$

Therefore, the likelihood function, which is the *a priori* joint probability of obtaining the actual results of the sample, is

$$L = \prod_{j=1}^{\tilde{J}} f_{k(j)} \times \prod_{j=\tilde{J}+1}^{J} \left[\sum_{k=k^{(m)}(j)+1}^{M} f_k \right] . \quad (3)$$

It is convenient to write $L = e^{-(1/2)S}$.

Denote by $N(k)$ the number of drawings in which $\alpha_j \leq \alpha_j^{(m)}$ and α_j falls into bin k. [$N(k)$ is simply the histogram of detected values of α.] Clearly one has

$$\sum_{k=1}^{M} N(k) = \tilde{J} \,.$$

Denote by $U(k)$ the number of drawings in which $\alpha_j > \alpha_j^{(m)}$ and $k^{(m)}(j) + 1 = k$. [$U(k)$ is simply the "histogram" of upper limits for those drawings in which α is not detected. $U(k)$ is the number of drawings in which α must fall in one of the bins whose label is k or larger. It is not an ordinary histogram.] Clearly one has

$$\sum_{k=1}^{M} U(k) = J - \tilde{J} \,.$$

With these definitions S can be written in the form

$$S = -2 \sum_{k=1}^{M} (N(k) \ln f_k) - 2 \sum_{k=1}^{M} \left[U(k) \ln \left(\sum_{k'=k}^{M} f_{k'} \right) \right] . \quad (4)$$

c) *Analytic Solution*

We now estimate the function $f(\alpha)$ using a nonparametric approach. All the different probabilities f_k are considered to be free parameters, and no functional form is presumed. The best estimate is obtained by maximizing L, or equivalently by minimizing S, with respect to the f_k's, subject to the constraint that f is normalized. We use the method of Lagrange multipliers. We introduce the constraint function and equation

$$g \equiv \sum_{k=1}^{M} f_k - 1 = 0 \,. \quad (5)$$

By minimizing the appropriate auxiliary function we find

$$f_n = \frac{N(n)}{J - \sum_{k=1}^{n} [U(k)/(1 - \sum_{k'=1}^{k-1} f_{k'})]} . \quad (6)$$

[For $n = 1$ and $k = 1$, the quantity in the square brackets is $U(1)$.]

Equation (6) is an analytic, recursive solution for the

[5] More generally, k may be allowed to run from 1 to ∞ or from $-\infty$ to $+\infty$, or from $-\infty$ to -1.

values of f_n. To see this note that the sum over k runs from 1 to n, so for each term $k \leq n$. For a given k, the sum over k' runs from 1 to $k - 1$, *so for each term* $k' < k$. Therefore $k' < n$. Thus equation (6) expresses f_n in terms of known numbers $[J, N(n), U(k)]$ and of values of $f_{k'}$ with $k' < n$. It is therefore possible to calculate f_1, then f_2, then f_3, and so on until all f_n's are found.

Thus we have shown that our method yields a simple analytic solution for the best estimate of a distribution function of one independent variable, when the range of values of this variable is binned, in a nonparametric approach. Generalizations of our basic method are discussed in § V, including a description of error analysis for the shape of $f(\alpha)$ and for any physical quantity that depends on it.

We wish to remark that in pathological cases it may perhaps happen that not all values of f_n obtained with equation (6) turn out to be between 0 and 1.[6] In such cases one must minimize numerically the function S of equation (4) subject to the constraint (5) and to the constraints $0 \leq f_n \leq 1$ for $1 \leq n \leq M$.

d) *Intuitive Derivation*

Equation (6) can be understood in terms of very simple intuitive arguments. In an estimation problem where there are no thresholds and all the α_j's are *a priori* guaranteed to be detected, the best estimates for the probabilities f_n are simply $N(n)/J$, i.e., the number of α's detected in each bin divided by the total number of drawings. In our estimation problem some α's are not detected. Consider those drawings in which α is not detected and for which α must fall into one of the bins with index k or larger. The number of these drawings is $U(k)$. What is the most likely distribution of the actual $U(k)$ values of α? The conditional probability that α falls into bin n, given that it has to fall into bins with index k or larger, is

$$f_n \Big/ \sum_{k'=k}^{M} f_{k'}$$

for $n \geq k$. The $U(k)$ values of α will therefore be distributed on the average in such a way that bin n (with $n \geq k$) contains

$$U(k) f_n \Big/ \sum_{k'=k}^{M} f_{k'}$$

values. We can thus define an "effective" number of detected α's in bin n given by

$$N_{\text{eff}}(n) = N(n) + \sum_{k=1}^{n} \frac{U(k) f_n}{\sum_{k'=k}^{M} f_{k'}} . \tag{7}$$

After "distributing" the undetected α's among the various bins, we use the result for the estimation problem without thresholds, namely,

$$f_n = \frac{N_{\text{eff}}(n)}{J} . \tag{8}$$

Equations (7) and (8) lead immediately to our formal solution, equation (6). We also note that in the special case where all the thresholds $\alpha_j{}^m$ are very large and all the α_j's are detected, $U(k) = 0$ for all bins, and equation (6) reduces to the simple result, $f_n = N(n)/J$.

[6] That is, if the true solution to the minimalization problem is not intrinsic to the M-dimensional cube $0 \leq f_n \leq 1$ but is a boundary solution.

TABLE 1

"Histograms" of Detected Values of $\alpha_{0,x}$ and of Lower Limits $\alpha_{0,x}{}^m$ for the Quasar Sample

k	$N(k)$	$U(k)$
1	2	0
2	1	1
3	4	1
4	4	0
5	3	1
6	6	0
7	3	3
8	1	1
9	2	0
10	1	1

IV. THE AVERAGE QUASAR L_x/L_{opt} RATIO

a) *Best Estimates*

The basic data obtained in the quasar survey with the *Einstein* observatory are described by Tananbaum *et al.* (1979) and in § II of this paper. The preliminary sample contains $J = 35$ quasars. The number of detected values of $\alpha_{0,x}$ is $\tilde{J} = 27$, while for the remaining $J - \tilde{J} = 8$ quasars lower limits $\alpha_{0,x}{}^m$ were obtained. All the detected values and lower limits are in the range from 0.94 to 1.86. We therefore consider the permissible range for $\alpha_{0,x}$ to be [0.9, 1.9].[7] We bin this range into $M = 10$ equal bins with width $\Delta\alpha_{0,x} = 0.1$. The bins are labeled by an index k with $1 \leq k \leq 10$. The histogram $N(k)$ of detected values of $\alpha_{0,x}$ in each bin and the "histogram" $U(k)$ of lower limits for $\alpha_{0,x}$ (as defined in § III*b*) can be obtained from Figure 1. These "histograms" are summarized in Table 1.

The best estimate for the set of probabilities f_k that describes the luminosity function $f(\alpha_{0,x})$, can be derived recursively from equation (6). The resulting solution is given in Figure 2.

The best estimate for the average $\langle L_{2\,\text{keV}}/L_{2500\,\text{Å}} \rangle$ is derived from the above solution using equation (2), which after binning reads

$$\langle L_{2\,\text{keV}}/L_{2500\,\text{Å}} \rangle = \sum_{k=1}^{M} f_k \,\text{dex}\,(-2.605\bar{\alpha}_k) , \tag{9}$$

where $\bar{\alpha}_k$ is the central value of $\alpha_{0,x}$ in bin k (i.e., $\bar{\alpha}_1 = 0.95$, $\bar{\alpha}_2 = 1.05$, etc.). The resulting value is $\langle L_{2\,\text{keV}}/L_{2500\,\text{Å}} \rangle = 5.16\ 10^{-4}$. We can express this value by defining an "effective" value for $\alpha_{0,x}$ using an

[7] But see also below.

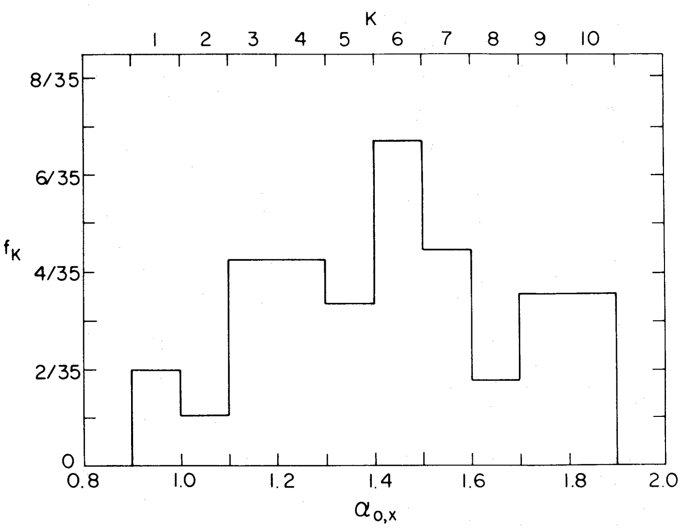

FIG. 2.—The best estimate of the luminosity function $f(\alpha_{0,x})$ for quasars determined from the preliminary quasar sample, using the nonparametric approach.

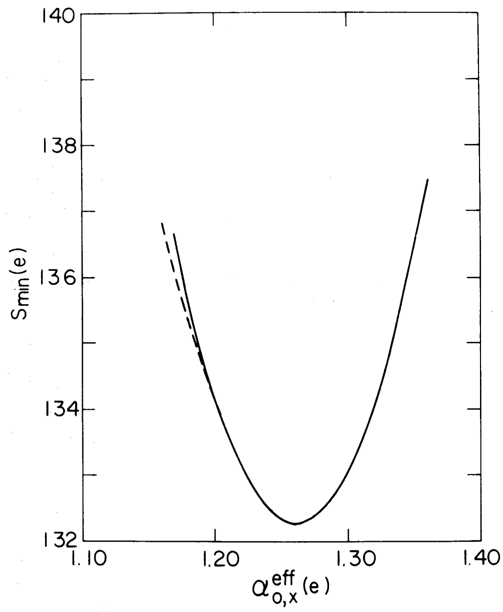

FIG. 3.—The minimum value of the function S (related to the likelihood function L by $L = e^{-(1/2)S}$), as a function of the trial value of $\alpha_{0,x}{}^{\text{eff}}$, from the preliminary quasar sample. *Solid line*, with 10 bins of $\alpha_{0,x}$ spanning the range [0.9, 1.9]. *Broken line*, with 12 bins of $\alpha_{0,x}$ spanning the range [0.8, 2.0]. The two curves coincide for $1.205 \leq \alpha_{0,x}{}^{\text{eff}} \leq 1.36$.

expression analogous to equation (1), namely,

$$\langle L_{2\,\text{keV}}/L_{2500\,\text{Å}} \rangle = \text{dex}\,(-2.605\alpha_{0,x}{}^{\text{eff}})\,. \qquad (10)$$

We obtain $\alpha_{0,x}{}^{\text{eff}} = 1.26$.

b) *Range of Uncertainty*

We now calculate the statistical range of uncertainty for $\alpha_{0,x}{}^{\text{eff}}$. Let us denote by e the value of $\langle L_{2\,\text{keV}}/L_{2500\,\text{Å}} \rangle$. We choose a trial value of e, and calculate for it the minimum value of the function S (of eq. [4]) subject to the normalization constraint (of eq. [5]) and subject to the further constraint that the right-hand side of equation (9) be numerically equal to e.[8] The resulting function $S_{\text{min}}(e)$ is plotted in Figure 3 (solid line), where each trial value of e is represented by the corresponding trial value of $\alpha_{0,x}{}^{\text{eff}}$. The overall minimum of this function S_{min} is at $\alpha_{0,x}{}^{\text{eff}} = 1.26$, which verifies our analytic solution for the best estimate. As we explain below in § V*a*, the 68% (1 σ) confidence limits for $\alpha_{0,x}{}^{\text{eff}}$ are obtained by requiring that $S_{\text{min}}(e) - S_{\text{min}} \leq 1$; they are therefore (1.217, 1.306). The 95% (2 σ) confidence limits correspond to $S_{\text{min}}(e) - S_{\text{min}} \leq 4$; they are therefore (1.174, 1.350). In other words, the 2 σ estimate for $\alpha_{0,x}{}^{\text{eff}}$ is 1.26 ± 0.09.

c) *Dependence on the Permissible Range of* $\alpha_{0,x}$

We have assumed so far that the permissible range of $\alpha_{0,x}$ is [0.9, 1.9], as all the detected values of $\alpha_{0,x}$ and all the lower limits, in the preliminary sample, fall in that range. It is in principle possible that the luminosity function $f(\alpha_{0,x})$ has nonzero small probabilities outside of the above range. We therefore discuss now how our results will change if we include additional bins of $\alpha_{0,x}$ in the analysis; namely, bins labeled by values of $k \leq 0$ that correspond to $\alpha_{0,x} < 0.9$, and bins labeled by values of $k \geq 11$ that correspond to $\alpha_{0,x} > 1.9$.

The first important point to notice is that the best estimate for the shape of $f(\alpha_{0,x})$ will not change, and consequently the best estimate for $\langle L_{2\,\text{keV}}/L_{2500\,\text{Å}} \rangle$ will not change. This follows directly from our analytic solution (eq. [6]). All the additional bins contain no detected values of $\alpha_{0,x}[N(n) = 0$ for $n \leq 0$ and for $n \geq 11]$. Thus $f_n = 0$ for all such "empty" bins, and the "new" values of f_n for $1 \leq n \leq 10$, will now be equal to the "old" values obtained previously without the added bins.

The range of uncertainty for $\langle L_{2\,\text{keV}}/L_{2500\,\text{Å}} \rangle$ can, however, become wider. The new value of $S_{\text{min}}(e)$ is calculated using more free parameters than the old value; therefore, it may be lower than the old value. Thus more values of e may enter the new confidence interval for any specified confidence level.

To study this possibility numerically, we have recalculated the function $S_{\text{min}}(e)$ adding two new bins of $\alpha_{0,x}$. One new bin is labeled by $k = 0$ and corresponds to the range [0.8, 0.9] of $\alpha_{0,x}$; the other new bin is labeled by $k = 11$ and corresponds to the range [1.9, 2.0]. The "new" function $S_{\text{min}}(e)$ is plotted in Figure 3 (broken line), together with the "old" $S_{\text{min}}(e)$ (solid line). We deduce that the 1 σ limits for $\alpha_{0,x}{}^{\text{eff}}$ and the 2 σ upper limit for $\alpha_{0,x}{}^{\text{eff}}$ did not change by adding the two new bins, but the 2 σ lower limit for $\alpha_{0,x}{}^{\text{eff}}$ did change from 1.174 to 1.168.

It is easy to understand intuitively the trends in these numerical results. From equations (9) and (10) it

[8] This minimalization problem was solved numerically using routine E04WAF of the NAG library on the VAX computer of the Center for Astrophysics. The calculation of one value of $S_{\text{min}}(e)$ took less than 5 s of computer time.

follows that $\alpha_{0,x}{}^{\text{eff}}$ is sensitive numerically only to the probabilities f_k corresponding to the smallest allowed values of $\alpha_{0,x}$. Thus adding new bins at larger values of $\alpha_{0,x}$ has almost no effect on the calculated ranges of uncertainty. Adding new bins at smaller values of $\alpha_{0,x}$ has no effect on the calculated upper limit of $\alpha_{0,x}{}^{\text{eff}}$ but has a noticeable effect of reducing the calculated lower limit of $\alpha_{0,x}{}^{\text{eff}}$.

It follows from our general arguments and from our numerical results that the calculated range of uncertainty for $\alpha_{0,x}{}^{\text{eff}}$ is not unique. This is a common difficulty for all nonparametric methods whenever the permissible range of the variable α is not bounded by some additional information and one attempts to estimate an unbounded function of α. To overcome this difficulty one assumes that the range of permissible values of α is not much wider than the range dictated by the results of the sample. This is justified particularly if the results of the sample indicate that $f(\alpha)$ indeed approaches zero toward the relevant edges of that range.

In our case, the best estimate we found for $f(\alpha_{0,x})$, given in Figure 2, indicates that $f(\alpha_{0,x})$ becomes quite small toward the lower edge of the range of $\alpha_{0,x}$ dictated by the sample. Furthermore, our numerical results show that by extending the lower edge of the permissible range from 0.9 to 0.8 [where the extrapolation of the best-estimate $f(\alpha_{0,x})$ crosses zero], the 2 σ lower limit of $\alpha_{0,x}{}^{\text{eff}}$ was reduced by a rather small amount of 0.006. We conclude that a reasonable and realistic estimate for $\alpha_{0,x}{}^{\text{eff}}$, and for its 2 σ uncertainty, based on the results of the preliminary sample, is 1.26 (+0.09, −0.10). This result is numerically sensitive to the few quasars with smallest $\alpha_{0,x}$ values. This result is also possibly subject to the biases discussed in § II.

V. GENERALIZATIONS

We present in this section several generalizations of our method, to complement the description of its basic version given in § III. These generalizations include error estimates for the shape of the luminosity function $f(\alpha)$ and for any physical quantity that depends on it; a formulation of the method without binning the range of values of α; a parametric variant of the method, where a functional form is presumed; and the estimation of luminosity functions of several variables.

We follow closely the notations introduced in § III, and the reader is referred to our discussion there for the definitions of the various symbols.

a) Error Estimates

The probabilities f_k that characterize the shape of the distribution function $f(\alpha)$ are estimated by a maximum-likelihood method. Therefore, standard $\Delta\chi^2$ techniques, applied on the function S, can be used to find the uncertainties of these parameters.

In many cases of astrophysical interest one wishes to find the range of uncertainty of some particular physical quantity that depends on the shape of the luminosity function. In our case we wish to calculate the uncertainty of $\langle L_{2\,\text{keV}}/L_{2500\,\text{Å}}\rangle$ that depends on the f_k's through equation (9). Such error estimates can be done in practice as follows.

Let e be a physical quantity that depends on the values of f_k through a function $E(f_k)$. Initially e is not a parameter of the distribution. We can make e such a parameter by performing a transformation of variables. One then estimates the uncertainty in e by treating it as a "single interesting parameter" (see Avni 1976, 1978). By transforming back to the original variables, we find that the confidence interval for the quantity e is simply and practically given as the set of values of e that satisfy

$$S_{\min}\begin{pmatrix}\text{constrained by}\\ E(f_k)=e \text{ and } \sum f_k = 1\end{pmatrix} - S_{\min}\begin{pmatrix}\text{constrained by}\\ \sum f_k = 1\end{pmatrix} \le \Delta\,, \quad (11)$$

where, e.g., $\Delta = 1$ for 68%, $\Delta = 2.71$ for 90%, $\Delta = 4$ for 95%. The minimization problem in the first term of equation (11) must in general be solved numerically. It is a rather simple problem using available standard computer routines.

b) Formulation without Binning

Our method can be formulated without binning the range of the variable α into finite-size bins. The *a priori* probability that in drawing j the value of α will be smaller than its threshold and that it will be within an infinitesimal range $(d\alpha)_j$ around α_j is $f(\alpha_j)(d\alpha)_j$. The *a priori* probability that in drawing j the value of α will be larger than its threshold is $\int_{\alpha_j^{(m)}}^{\infty} d\alpha f(\alpha)$. The function S therefore obtains the form (cf. eq. [3])

$$S = -2\sum_{j=1}^{\tilde{J}} \ln f(\alpha_j) - 2\sum_{j=\tilde{J}+1}^{J} \ln \int_{\alpha_j^{(m)}}^{\infty} d\alpha f(\alpha)\,, \quad (12)$$

where we have already subtracted from S a constant term,

$$-2\sum_{j=1}^{\tilde{J}} \ln (d\alpha)_j\,,$$

which plays no role in the minimization process. The normalization constraint is expressed by the following equation (cf. eq. [5])

$$g \equiv \int d\alpha f(\alpha) - 1 = 0\,. \quad (13)$$

In the nonparametric approach, we minimize S with respect to the function f subject to the normalization constraint, using the method of Lagrange multipliers. We obtain the following solution:

$$f(\alpha) = \sum_{j=1}^{\tilde{J}} \left\{\frac{\delta(\alpha-\alpha_j)}{J - \sum_{j'=\tilde{J}+1}^{J}[\theta(\alpha_j - \alpha_{j'}{}^{(m)})/\int_{\alpha_{j'}^{(m)}}^{\infty} d\alpha' f(\alpha')]}\right\}, \quad (14)$$

where δ is the familiar delta-function, and θ is the step-function [$\theta(x) = 1$ for $x \geq 0$, $\theta(x) = 0$ for $x < 0$]. Equation (14) expresses $f(\alpha)$ as a sum of delta-functions concentrated at the detected values of α, namely, at the $\tilde{J}$-values α_j. It is therefore more meaningful to express the solution in terms of the cumulative distribution $F(\alpha)$ defined by

$$F(\alpha) = \int^{\alpha} d\alpha' f(\alpha') . \tag{15}$$

By integrating equation (14) we get, for α not equal to any of the α_j's, the following solution:

$$F(\alpha) = \sum_{1 \leq j \leq \tilde{J},\, \alpha_j < \alpha} \times \left[J - \sum_{\tilde{J}+1 \leq j' \leq J,\, \alpha_{j'}^{(m)} < \alpha_j} \frac{1}{1 - F(\alpha_{j'}^{(m)})} \right]^{-1} . \tag{16}$$

Here, $F(\alpha)$ has discrete jumps at the values of detected α_j's. For a given value of α, the sum over j in equation (16) extends only over drawings for which $\alpha_j < \alpha$; for a fixed j the sum over j' extends only over drawings for which $\alpha_{j'}^{(m)} < \alpha_j$; hence $\alpha_{j'}^{(m)} < \alpha$. Thus equation (16) is a recursive solution for $F(\alpha)$.

Our basic estimation problem can therefore be formulated without binning (eq. [12] and [13]). In the particular case when the best estimate for the distribution $f(\alpha)$ is calculated nonparametrically, we have obtained an analytic recursive solution (eq. [16]). The nonparametric best estimate for any physical quantity that depends on $f(\alpha)$ is directly obtained by using that solution. Such estimates are free from the small ambiguity introduced by any choice of finite-size bins. We note, however, that binning is practically required in order to calculate the range of uncertainty for any such physical quantity.

c) Parametric Approach

When a particular functional form of $f(\alpha)$ is presumed, the function S (of eq. [4] or eq. [12]) and the constraint function g (of eq. [5] or eq. [13]) are considered to be functions of the parameters of the distribution. All standard maximum-likelihood methods for parameter estimation can be applied. The new ingredient which we introduce in this case is that the likelihood function incorporates lower limits for undetected α's, in addition to the detected values of α. The analysis therefore makes use of both flux measurements and flux upper limits.

d) Functions of Several Variables

Our method can be generalized to facilitate numerical calculations of distribution functions of several variables [e.g., the bivariate quasar luminosity function $f(L_x/L_{\text{opt}}, L_{\text{rad}}/L_{\text{opt}})$, using an appropriate sample of optically selected quasars].

When f is a function of D variables, "α" is a vector $\boldsymbol{\alpha}$ of D dimensions, and so are the indices $\boldsymbol{k}$, the detected values $\boldsymbol{\alpha}_j$, and the lower limits $\boldsymbol{\alpha}_j^{(m)}$. The bins into which the range of $\boldsymbol{\alpha}$ is divided are of D dimensions. The "histogram" $N(k)$ is now a D-dimensional matrix $N(\boldsymbol{k})$, as is the bin probability $f_{\boldsymbol{k}}$. Objects in the sample can be either (i) detected at all the D variables ($\tilde{J}$ such objects), or (ii) detected at some of the variables and undetected (possess only lower limits) at the other variables, or (iii) undetected at all variables.

The function L corresponding to the case when the range of $\boldsymbol{\alpha}$ is binned is still given by equation (3); the summation

$$\sum_{k = k^{(m)}(j)+1}^{M}$$

is now understood to mean a summation over all bins into which $\boldsymbol{\alpha}_j$ can fall in cases (ii) and (iii). The function S corresponding to the case when the range of $\boldsymbol{\alpha}$ is not binned is still given by equation (12); the integral

$$\int_{\alpha_j^{(m)}}^{\infty} d\alpha$$

is now understood to mean an integral over the region of $\boldsymbol{\alpha}$ inside which $\boldsymbol{\alpha}_j$ can be in cases (ii) and (iii). The normalization constraints are still given by equations (5) and (13); the summation and integral are now over the whole *a priori* permissible range of $\boldsymbol{\alpha}$.

The derivation of best estimates and ranges of uncertainty for the distribution function $f(\boldsymbol{\alpha})$, and for any physical quantity that depends on it, involves again the minimization of S with respect to f subject to the appropriate constraints. This can be done numerically in precisely the same way as for the simpler case of a one-dimensional variable. Both parametric and nonparametric approaches can be utilized. The only difference is that for distribution functions of more than one variable, no analytic solution for the best estimate of f can be derived in general. This is so because there is no "natural" ordering of the variable $\boldsymbol{\alpha}$ that would allow for a recursive solution.

VI. SUMMARY

We have derived a method for determining luminosity functions from preselected random samples. The method incorporates flux measurements of detected objects, as well as flux upper limits for undetected objects. The method yields best estimates and ranges of uncertainty for the shape of the luminosity function and for any physical quantity that depends on it.

Our method can be formulated nonparametrically, without any prior assumption on the shape of the luminosity function. Alternatively, the method can be formulated parametrically, when a particular functional form is presumed. The method can be applied both when the range of values of the random variable is binned, and also when no binning is employed. The method can be applied to determine luminosity functions that depend on any number of variables.

The basic ingredient of the method is the likelihood function L, expressed as $L = e^{-(1/2)S}$, where S is given by equations (4) or (12), and by their generalizations described in § V. The likelihood function is constructed using both ordinary measured fluxes and also flux upper limits. Best estimates for the luminosity function and for physical quantities depending on it are obtained by minimizing S subject to the constraint that the luminosity function be normalized (eqs. [5] and [13], and their generalizations described in § V). In the particular case where one estimates a function of one variable in the nonparametric approach, the problem is solved analytically, yielding a simple recursive expression for the luminosity function (eq. [6] or eq. [16]). Ranges of uncertainty for the luminosity function and for relevant physical quantities are obtained by minimizing S subject to the normalization constraint, and subject to an additional constraint that holds fixed a trial-value of the physical quantity under consideration (eq. [11] and its generalizations described in §§ V*c* and V*d*). These minimization problems are solved numerically rather easily with available computer routines.

We applied our method to a preliminary sample of quasars observed with the *Einstein* observatory. We determined the best estimate of the power-law index $\alpha_{0,x}^{\text{eff}}$ that characterizes the average value $\langle L_{2\,\text{keV}}/L_{2500\,\text{Å}} \rangle$ (see eqs. [1] and [10], and the beginning of § II). We have also determined the statistical uncertainty for $\alpha_{0,x}^{\text{eff}}$. The 2 σ limits of $\alpha_{0,x}^{\text{eff}}$ depend weakly on the assumed permissible range of $\alpha_{0,x}$. Since the estimated distribution of $\alpha_{0,x}$ indicates that this permissible range does not extend significantly below $\alpha_{0,x} \lesssim 0.9$, we determine a 2 σ estimate of $\alpha_{0,x}^{\text{eff}} = 1.26$ $(+0.09, -0.10)$. Possible systematic biases in the sample are discussed; these possible biases will be resolved by further observations currently in progress.

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