Stellar Structure

Chapter 15

Stellar Structure

We know external properties of a star

L, M, R, T_{eff}, (X, Y, Z)

From this, can we infer the internal structure?

Apply basic physical principles

Stellar Structure

Overall Picture: Stars in Equilibrium -gravity balanced by pressure -if not collapse or expand on very short time scales - eg remove pressure entirely Sun collapses on free-fall timescale.

Free-fall from: $s = ut + 1/2gt^2$ ($u=0$, $s=R_{sun}$, $g=GM_{sun}/R_{Sun}{}^2$)
Thus $t_{ff} = (2R_{sun}{}^3/GM_{sun})^{1/2}$

with $R=7 \times 10^8$ m, $G=6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, $M=2 \times 10^{30}$ kg get
t~ 40 minutes!!

Pressure here is gas pressure - other sources (rotation, magnetic, radiation) much less important in Sun.

Stellar Forces

Gravity and Gas Pressure

Hydrostatic equilibrium (§ 15.1) - balance between gravity and gas pressure

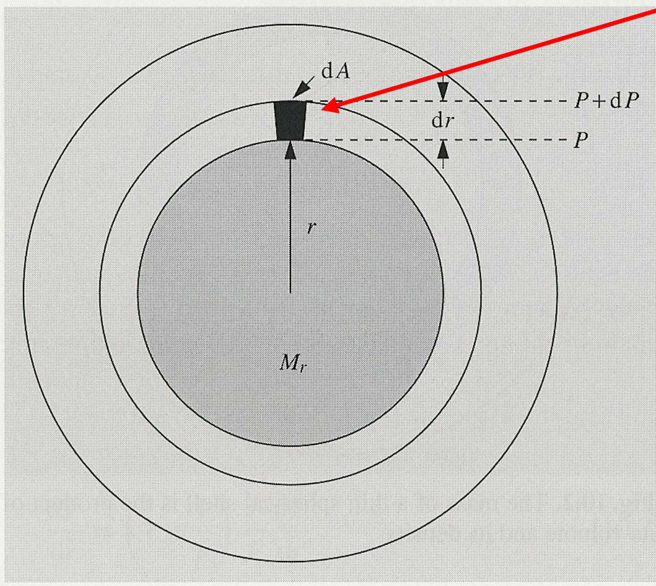


Fig. 10.1. In hydrostatic equilibrium the sum of the gravitational and pressure force acting on a volume element is zero

Consider small cylinder located at radius, r, from centre of the star:

- P + dP pressure on top of cylinder, P on bottom
- dr is height cylinder, dA its area and dm its mass
- Volume of the cylinder is dV = dAdr
- Mass of the cylinder is dm = ρdAdr where
- ρ = ρ(r) is the gas density at the radius r
- The total mass inside radius r is M_r
- Gravitational force on volume element is

$dF_g = -GM_r dm/r^2 = -GM_r \rho dAdr/r^2$ (- as force directed to centre of star)

Stellar Forces

Gravity and Gas Pressure

Hydrostatic equilibrium (§ 10.1) - balance between gravity and gas pressure

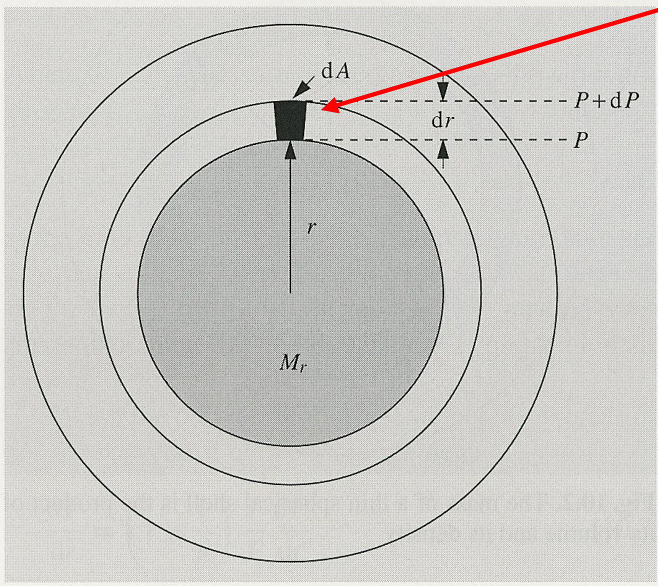


Fig. 10.1. In hydrostatic equilibrium the sum of the gravitational and pressure force acting on a volume element is zero

Consider small cylinder located at radius, r, from centre of the star:

- Gravitational force on volume element is $dF_g = -GM_r dm/r^2 = -GM_r \rho dAdr/r^2$ (- as force directed to centre of star)
- Net pressure force acting on element is $dF_p = PdA - (P + dP)dA = -dPdA$ (dP is negative as pressure decreases outward)
- Equilibrium condition: the total force acting on volume is zero i.e.

$$0 = dF_g + dF_p = -GM_r \rho dAdr/r^2 - dPdA \text{ or}$$

1. $dP/dr = - GM_r \rho/r^2$ (Equation of Hydrostatic Equilibrium)

Stellar Forces

Is Sun in hydrostatic equilibrium?

- Let $ma = F_g - F_p$ (m mass, a acceleration)
- Let $F_p = 1.00000001\ F_g = (1 + 1 \times 10^{-8})F_g$
- Then acceleration at solar surface given by:

  $ma = F_g - F_p = F_g(1 - 1 - 1 \times 10^{-8}) = -10^{-8}F_g$

  $= -GM_{sun}m(10^{-8})/R_{sun}^2$

  m cancels and putting in numbers

  $a = (270\ m\ s^{-2})(10^{-8}) = 2.7 \times 10^{-6}\ m\ s^{-2}$

- Displacement of solar surface in t = 100 days (8.6×10^6 s) would be:

  $d = 1/2\ at^2 = 1.0 \times 10^8\ m = 0.14$ solar radii!

- So if equilibrium is unbalanced by only 1 part in 10^8, Sun would grow (or shrink) by ~14% in a few months - clearly not observed
- 14% change in radius would result in a 30% change in L resulting in 7% change in global temperature of Earth!!

Stellar Forces

Are other forces important?

Rotation (gives centripetal and coriolis forces)

- $F_{cent} = m\,\Omega^2 r$ where Ω = angular velocity (radian s^{-1})
- At equator on stellar surface, $F_{cent} = m\,\Omega^2 R_*$ and $F_g = GM_* m/R_*^2$
- Thus, $F_{cent}/F_g = \Omega^2 R_*^3/GM_*$
- e.g. Sun: $R = 7 \times 10^8$ m, $P_{rot} = 25$ days
  $\rightarrow \Omega_{sun} = 3 \times 10^{-6}$ rad s^{-1},
  $M_{sun} = 2 \times 10^{30}$ kg
  $\rightarrow F_{cent}/F_g = 2 \times 10^{-5}$
- For F_{cent} to be important, a star must be large and rotating rapidly. There are some examples of such stars. e.g. Be stars which show emission lines.

Stellar Forces

Are other forces important?

Radiation Pressure

- Before we do this let's estimate pressure at centre of Sun
- Crudely, $dP/dr \sim \Delta P/\Delta r \sim (P_s - P_c)/(R_s - R_c)$ where P_c is the central pressure, P_s is the surface pressure (= 0), R_s is the surface radius and R_c the radius at center = 0)
- So $dP/dr \sim (0 - P_c)/(R_s - 0)$ or $dP/dr \sim - P_c / R_s$
- Now apply to the Sun. Substituting into Hydrostatic equilibrium eq., $dP/dr = - GM_r\rho/r^2$, gives

$$-P_c/R_{sun} = -GM_{sun}\rho_{sun}/R_{sun}^2; \text{ or } P_c = GM_{sun}\rho_{sun}/R_{sun}$$

$(\rho_{sun} = 1400 \text{ kg m}^{-3}; R_{sun} = 7 \times 10^8 \text{ m})$

$\rightarrow P_c = 2.7 \times 10^{14} \text{ N m}^{-2}$ (or Pa) (Surface Earth 10^5 Pa)

Stellar Forces

Are other forces important?

Radiation Pressure

- The value of $P_c = 2.7 \times 10^{14}$ N m^{-2} (Pa) for the pressure at the Sun's centre is crude and too low by about 2 orders of magnitude. It has not taken into account the increased density near the Sun's centre. Theoretical models give a value of 2.5×10^{16} Pa
- Radiation pressure, $P_{rad} = (1/3)aT^4$ with $a = 7.57 \times 10^{-16}$ J m^{-3} K^{-4}
- At the Sun's centre, $T \sim 10^7$ K $\rightarrow P_{rad} = 7.6 \times 10^{12}$ N m^{-2} (Pa)
- This is a crude estimate but indicates that in stars like the sun, radiation pressure not important (compare with gas pressure 2.5×10^{16} Pa) - but it is important for hotter stars.

Stellar Structure

Are other forces important?

Magnetic pressure

- $P_{mag} = H^2/8\pi$ where H = field strength in Gauss, P in cgs units (dynes)
- 1 Gauss (cgs unit) = 10^{-4} Tesla
- At the centre of the Sun we'll see that:

  $P_{gas} \sim 2.5 \times 10^{16}$ Pa (1 Pa = 10 dyne/cm^2), so if mag pressure 0.1%

  $\rightarrow$ we need $\sim 10^7$ Tesla at centre for P_{mag} to be important
- At base of photosphere ("surface of Sun"):

  $P_{gas} \sim 10^4$ Pa

  $\rightarrow$ need fields of ~ 10 Tesla for P_{mag} to be important

  Measured value of P_{mag} at the surface is ~ 10^{-4} Tesla

- However, there are stars with surface fields of many kG and even giga G (magnetic white dwarfs)

Bottom line: For normal stars like the Sun, the only force we need to consider, in the first approximation, is the force due to gas pressure.

Mass Conservation, Energy Production

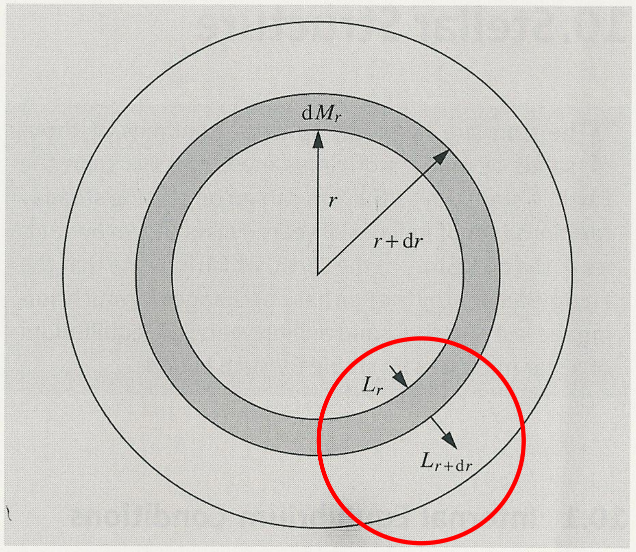


Fig. 10.3. The energy flowing out of a spherical shell is the sum of the energy flowing into it and the energy generated within the shell

Mass of shell of thickness dr at radius r is

$dM(r) = \rho \times V = \rho \times 4\pi r^2 dr \rightarrow$

2. $dM(r)/dr = 4\pi r^2 \rho(r)$ - Equation Mass Conservation

Luminosity produced by shell of mass dM is $dL = \varepsilon dM$,

ε = total energy produced /mass/sec by all fusion nuclear reactions and gravity

$\therefore dL(r) = \varepsilon 4\pi r^2 \rho(r) dr \rightarrow$

3. $dL(r)/dr = 4\pi r^2 \rho(r) \varepsilon(r)$ - Energy Production Equation

Here $\varepsilon(r) > 0$ only where $T(r)$ is high enough to produce nuclear reactions

In Sun, $\varepsilon(r) > 0$ when $r < 0.2\ R_{sun}$

Summary (So Far) Stellar Structure Equations

1. $dP/dr = -GM_r\rho(r)/r^2$	Equation of Hydrostatic Equilibrium
2. $dM(r)/dr = 4\pi r^2\rho(r)$	Equation of Mass Conservation
3. $dL(r)/dr = 4\pi r^2\rho(r)\varepsilon(r)$	Equation of Energy Production
4.	

Temperature Gradient

Energy Transport

- The fourth equation of stellar structure gives temperature change as function of radius r, i.e dT/dr.

- In the interior of stars like the Sun, conduction of heat (by electrons) is very inefficient as electrons collide often with other particles.

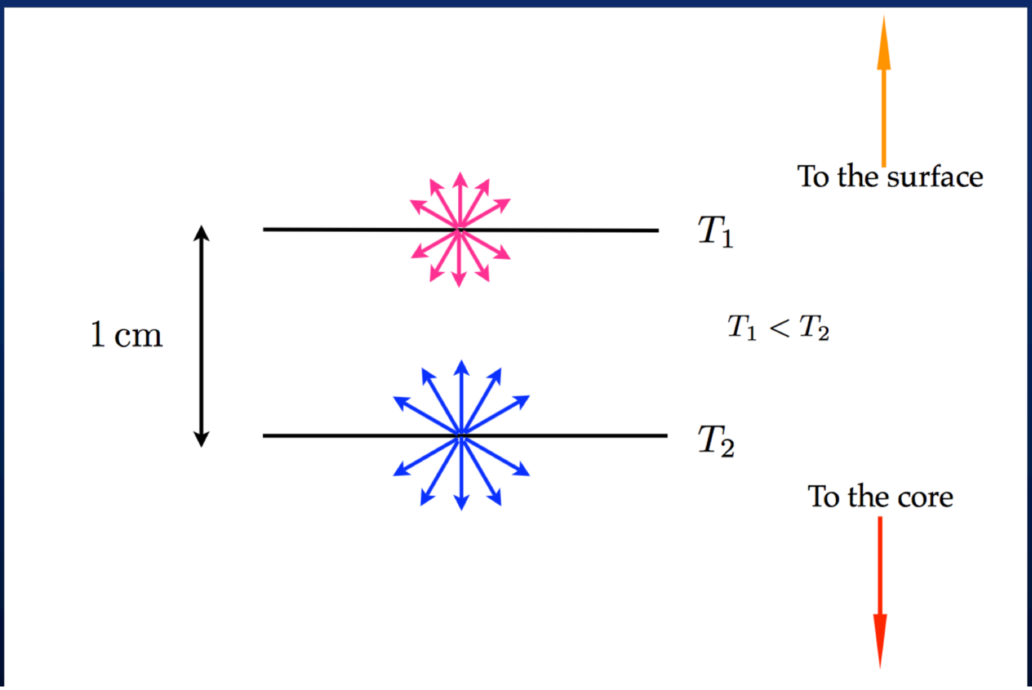
- However, in white dwarfs and neutron stars, heat conduction is a very important means of energy transport. In these stars, the mean free path of some electrons can be very long whereas the mean free path of their photons is extremely short.

Energy Transport

Energy Transport

➢Thus, the majority of energy is transported by radiation in the interior of most stars. Photons emitted in hot regions of a star are absorbed in cooler regions.

➢ Consider a small cell in the interior of a star with a volume of 1 cm^3 as seen in the simple diagram below.



Energy Transport

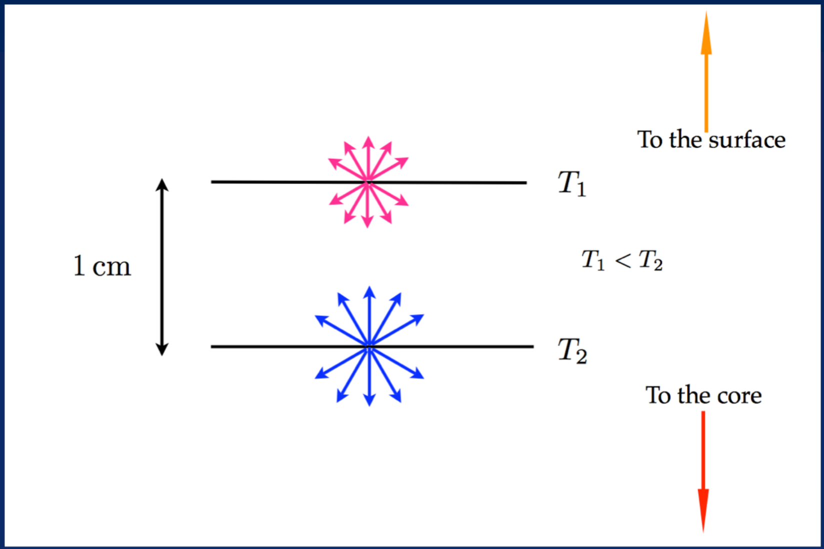
The flux from below is σT_2^4 (blackbody radiation)

The flux from above is σT_1^4

Hence the net flux through

the element is

$$F = \sigma(T_2^4 - T_1^4)$$



We can generalize this to $F = -d/dr(\sigma T^4)$

This flux needs to be multiplied by the "transparency" of the layer (some radiation will be absorbed) defined through κ so that $\kappa\rho dl$ gives fraction energy lost by absorption over distance dl ($\lambda\kappa\rho = 1$ makes sense as $\lambda=1/\kappa\rho$ is the mean free path) units κ are m^2/kg.

Energy Transport

Thus the total flux through the volume element is the net flux times the photon mean free path

$$F = -1/\kappa\rho \cdot d/dr(\sigma T^4)$$

We still need to equate the flux through a spherical surface of radius r to the total luminosity

$$F = L/4\pi\, r^2$$

This will give us $L = -1/\kappa\rho \cdot 4\pi\, r^2 \cdot d/dr(\sigma T^4)$

$$= -1/\kappa\rho \cdot 4\pi\, r^2 \cdot ac/4 \cdot 4T^3 \cdot dT/dr \ (1)$$

radiation constant (a) usually substituted for σ $a=4\sigma/c$

Which we can rearrange to $dT/dr = -1/ac \cdot \kappa\rho/T^3 \cdot L/4\pi\, r^2$

This is known as the equation of Radiative Equilibrium

Energy Transport

This equation gives the temperature gradient in a star required to carry the star's entire luminosity via radiation.

A star is said to be in radiative equilibrium when this holds.

We can use (1) to estimate the luminosity of the Sun

$L = -1/\kappa\rho \cdot 4\pi r^2 \cdot ac/4 \cdot 4T^3 \cdot dT/dr$ (1)

in MKS units (Watts)

$1/\kappa\rho \sim 0.001$m, $r=7\times10^8$m, $a=7.6\times10^{-16}$ Jm^{-3} K^{-4}, $T=5\times10^6$K we get the rough value of $\sim2.5 \times 10^{27}$ Watts compared with known value 4×10^{26} Watts - not bad!

Equation of State

Equation of State

- Expresses the dependence of P(pressure) on other parameters.
- Most common eqn. of state is the ideal gas law, $P = NkT$

  Where k = Boltzmann constant, N = # particles per unit volume, and T = temperature
- Holds at high accuracy for gases at low density.
- It is also accurate at high densities if the gas temperature is also high as in stellar interiors.
- Now we introduce the gas composition explicitly.
- Let X = mass fraction hydrogen in a star, Y same for He, and Z for everything else.
- (We have seen that $X = 0.73$, $Y=0.25$, and $Z = 0.02$.) Of course $X + Y + Z = 1$.

Equation of State

Equation of State

Now we tabulate the number of atoms and number of corresponding electrons per unit volume (here m_H is mass of the proton)

Element	*Hydrogen*	*Helium*	*Heavier*
No. atoms	$X\rho/m_H$	$Y\rho/4m_H$	$[Z\rho/Am_H]$ (usually small)
No. electrons	$X\rho/m_H$	$2Y\rho/4m_H$	$1/2\ AZ\rho/Am_H$

- Assume gas is fully ionized so sum all items to get:
  $N = (2X + (3/4)Y + (1/2)Z)\rho/m_H$
- Equation state then is:
  $P = (1/\mu)k\rho T/m_H$ with $1/\mu = 2X + (3/4)Y + (1/2)Z$ (# particles per unit mass)
- In the Sun, $1/\mu = 2(0.73) + 3/4(0.25) + 1/2(0.02) = 1.658$
  so $\mu = 0.60$ (μ is called the mean molecular weight) (eg μ for pure H gas?)

Summary Stellar Structure Equations

1. $dP/dr = - GM_r\rho(r)/r^2$	Equation of Hydrostatic Equilibrium
2. $dM(r)/dr = 4\pi r^2\rho(r)$	Equation of Mass Conservation
3. $dL(r)/dr = 4\pi r^2\rho(r)\varepsilon(r)$	Equation of Energy Production
4. $dT/dr = (-3/4ac)\ (\kappa\rho(r)/T^3)\ (L_r/4\pi r^2)$	Temperature Gradient
$P = (1/\mu)k\rho T/m_H$	Equation of State

- Plus boundary conditions:
  At centre - $M(r) \rightarrow 0$ as $r \rightarrow 0$; $L(r) \rightarrow 0$ as $r \rightarrow 0$
  At surface - $T(r), P(r), \rho(r) \rightarrow 0$ as $r \rightarrow R_*$
- These equations produce the Standard Solar Model and the Mass - Luminosity Relation

Standard Solar Model

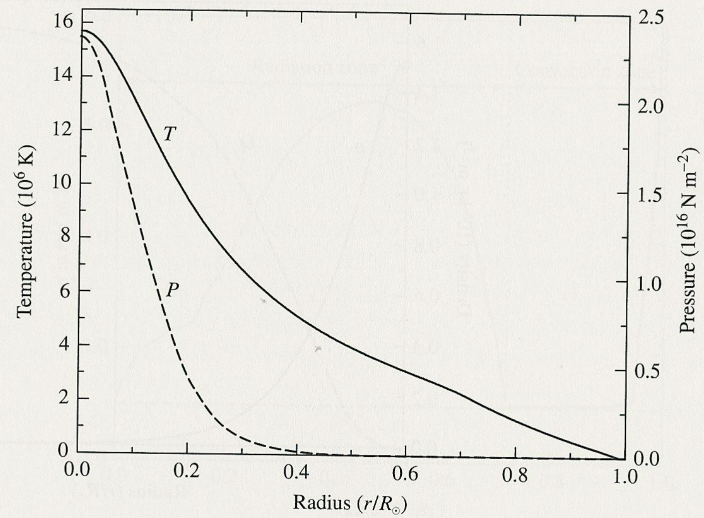


FIGURE 11.4 The temperature and pressure profiles in the solar interior. (Data from Bahcall, Pinsonneault, and Basu, *Ap. J.*, *555*, 990, 2001.)

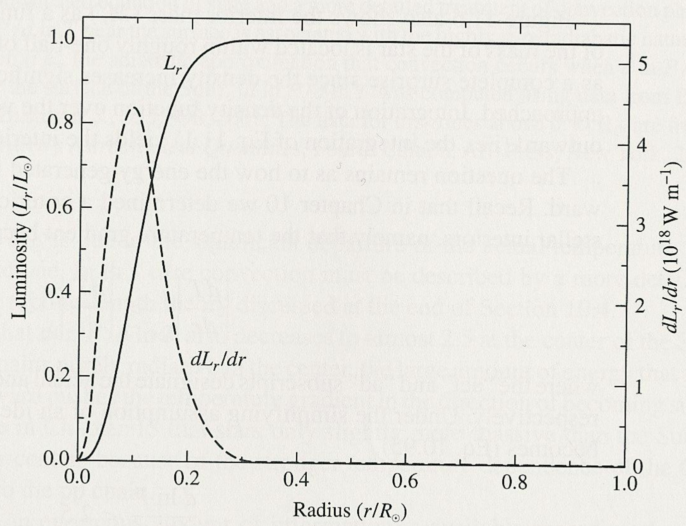
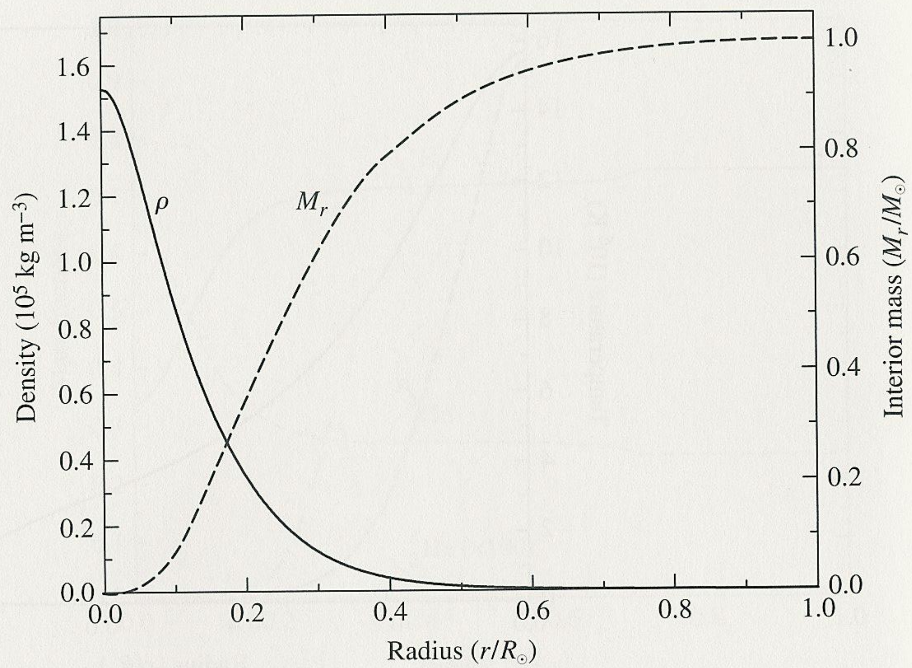


FIGURE 11.5 The interior luminosity profile of the Sun and the derivative of the interior luminosity as a function of radius. (Data from Bahcall, Pinsonneault, and Basu, *Ap. J.*, *555*, 990, 2001.)



Mass - Luminosity Relation

- Use the Equations of Stellar Structure to calculate how the luminosity of a star depends on its mass (the Mass Luminosity Relation).

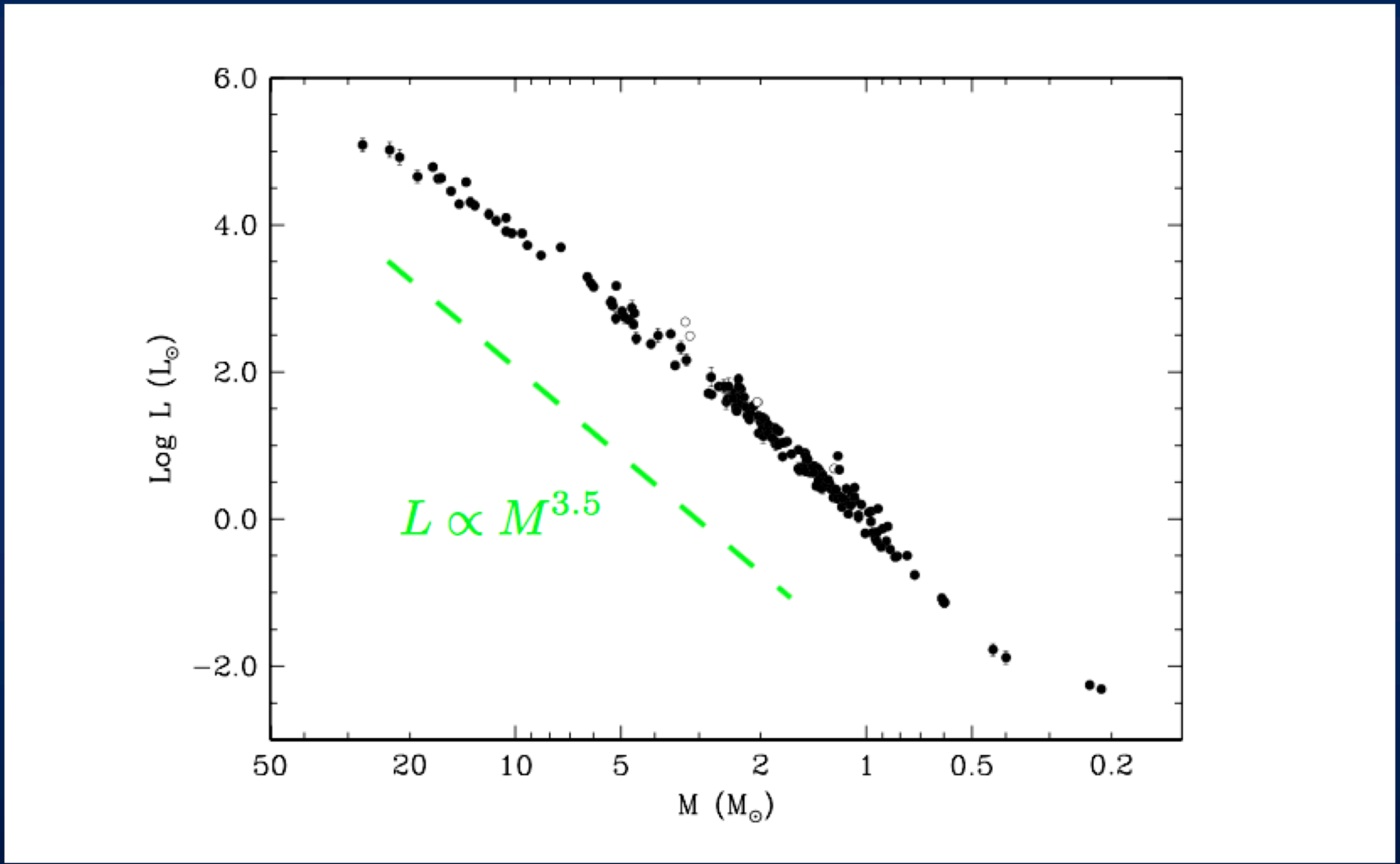
1. $dP/dr = - GM_r\rho(r)/r^2$	Equation of Hydrostatic Equilibrium
2. $dM(r)/dr = 4\pi r^2\rho(r)$	Equation of Mass Conservation
3. $dL(r)/dr = 4\pi r^2\rho(r)\varepsilon(r)$	Equation of Energy Production
4. $dT/dr = (-3/4ac)\ (\kappa\rho(r)/T^3)\ (L_r/4\pi r^2)$	Temperature Gradient
$P = (1/\mu)k\rho T/m_H$	Equation of State

- Density, $\rho \propto M/R^3$ ….eqn. (1)
- Substitute (1) into Hydrostatic Equil. Eqn., $dP/dr = - GM_r\rho(r)/r^2$

  to get $P \propto M^2/R^4$ …eqn. (2)
- Use Eqn. State, $P = (1/\mu)k\rho T/m_H$ with eqn. (2) is $T \propto M/R$ …eqn. (3)
- Put (2) and (3) into Radiative Equilibrium Eq.,

  $$dT/dr = (-3/4ac)\ (\kappa\rho(r)/T^3)\ (L_r/4\pi r^2)$$
- To get $L \propto M^3$ … eqn. (4) which is close to observed relationship, $L \propto M^{3.5}$

Observational Mass Luminosity Relationship



Central Temperature Sun

Use ideal gas law, $P = \rho kT/\mu m_H$

$\rightarrow P_c = \rho_c kT_c/\mu m_H$

$\rightarrow T_c = \mu m_H P_c/\rho_c k$

Approximate central density, ρ_c as $\langle\rho\rangle = 1400$ kg m^{-3}

Take $P_c = 2.7 \times 10^{14}$ Pa from earlier estimate, and $\mu = 0.60$

$\rightarrow T_c \sim 1.4 \times 10^7$ K (models predict about 1.6×10^7 K)

Agreement is fortuitous since the pressure estimate used as P_c and the density estimate, $\langle\rho\rangle$, used as ρ_c are both too low by a factor of 100.