3.1 Means and variances

The Poisson distribution has

$$p(n) = e^{-\mu}\frac{\mu^n}{n!}.$$

The mean is therefore

$$\begin{aligned} m &= \sum_{n=0}^{\infty} n\,p(n) && (1)\\ &= \sum e^{-\mu}\frac{\mu^n}{(n-1)!} && (2)\\ &= \mu e^{-\mu}\sum \frac{\mu^n}{n!} && (3)\end{aligned}$$

and the last term sums to e^{μ}.
A similar argument works for the variance.
For a power law, the upper and lower limits matter; suppose they are b and a. Within this range, the power law distribution is

$$p(x) = \frac{\gamma - 1}{a^{1-\gamma} - b^{1-\gamma}} x^{-\gamma}$$

and the mean is

$$m = \frac{(-a^{\gamma}b^2 + a^2 b^{\gamma})(-1+\gamma)}{(-a^{\gamma}b + ab^{\gamma})(-2+\gamma)}$$

which is quite a complicated expression. It becomes clearer in a few limits. If $\gamma = 3$, for instance, we have

$$m = 2a$$

showing that the bottom end determines the mean. If we have $\gamma = 2$, we can take limits to get

$$m = \frac{ab(\log[a] - \log[b])}{a - b}$$

showing that both a and b matter here. If $\gamma \to 1$ we get

$$m = \frac{a - b}{\log[a] - \log[b]}$$

and b dominates the mean.
The variance is intimidating:

$$\frac{-((-1+\gamma)(-a^{2\gamma}b^4 - a^4b^{2\gamma} + a^{3+\gamma}b^{1+\gamma}(-2+\gamma)^2 + a^{1+\gamma}b^{3+\gamma}(-2+\gamma)^2 - 2a^{2+\gamma}b^{2+\gamma}(3-4\gamma+\gamma^2)))}{((a^\gamma b - ab^\gamma)^2(-3+\gamma)(-2+\gamma)^2)}.$$

Taking some limits, we find (as expected) that we must have $\gamma < -4$ for the upper limit not to dominate the variance; also, we must have $\gamma > -1/2$ for the lower limit not to matter.

The variance of a Cauchy distribution is simple by comparison.

$$p(x) = \frac{1}{\pi(1+(x-\mu)^2)}$$

and clearly

$$\int (x-\mu)^2 p(x)$$

diverges linearly.